

The quasi-incompressible planet: some analytics

Zakir F. Seidov

Research Institute, College of Judea and Samaria, Ariel 44837, Israel

zakirs@yosh.ac.il

February 2, 2008

ABSTRACT

Exact and approximate analytical formulas are derived for the internal structure and global parameters of the spherical non-rotating quasi-incompressible planet. The planet is modeled by a polytrope with a small polytropic index $n \ll 1$, and solutions of the relevant differential equations are obtained analytically, to the second order of n . Some solved and unsolved problems of polytropic models are discussed as well.

Subject headings: planets: internal structure—quasi-incompressible model—polytropes

1. Introduction

Many classical problems of the theoretical astrophysics: internal structure, effects of rotation, tidal effects, pulsations and stability of the stars/planets are solved mainly in the case of incompressible (or homogeneous) liquid.

The point is that even the simplest (and basic) differential equation of internal structure - Lane-Emden equation (hereafter LEE) of index n is solved analytically only for three values of $n = 0, 1, 5$, each of them having their own deficiencies - case $n = 0$ corresponds to *incompressible* liquid, polytropic model with $n = 1$ has *constant* radius independent of its mass, and the star with $n = 5$ (and with $n > 5$) has *infinite* radius. Some 25 years ago SK (Seidov & Kuzakhmedov, 1977 (SK77), 1978 (SK78)) had presented the new analytical solutions of the LEE for index n only slightly differing from 0, 1, and 5, and also the series form of solution of LEE of arbitrary index. Several authors used and extended the approach of SK for the non-rotating and slowly rotating models - Seidov (1978a,b, 1979a,b); Seidov, Sharma & Kuzakhmedov (1979); Mohan & Al-Bayaty (1980); Jabbar (1984); Caimmi (1987); Horedt (1987, 1990); Medvedev & Rybicki (2001).

In this note I return to SK and present some analytical results for the *quasi-incompressible*

model of planet. The idea is to use the *perturbation theory* of differential equations: if we have the analytical solution of the ordinary differential equation (ODE) for some particular value of parameter $n = n_0$ (say $n_0 = 0$) then we may try to look for the analytical solution of the same ODE with $n \approx n_0$ (in our case $n \approx 0$).

2. Basic equation

The basic equation is LEE of index n :

$$\frac{1}{x^2} \frac{d}{dx} \left(x^2 \frac{dy}{dx} \right) = -y^n, \quad (1)$$

with initial conditions $y(0) = 1$, $y'(0) = 0$.

We look for solution $y(x)$ in an interval from $x = 0$ to the first zero $x = X$ such that $y(X) = 0$ (hereafter to be concise I use shorthand $s \equiv \sqrt{6}$).

Three classical analytical solutions of eq. (1) are long known (see e.g. (Chandrasekhar 1957)):

$$n = 0, \quad y = 1 - \frac{1}{6} x^2, \quad X = s, \quad \mu = 2s, \quad \rho_c/\rho_m = 1; \quad (2)$$

$$n = 1, \quad y = \frac{\sin x}{x}, \quad X = \pi, \quad \mu = \pi, \quad \rho_c/\rho_m = \pi^2/3; \quad (3)$$

$$n = 5, \quad y = (1 + \frac{1}{3} x^2)^{-1/2}, \quad X \rightarrow \infty, \quad \mu = \sqrt{3}, \quad \rho_c/\rho_m \rightarrow \infty. \quad (4)$$

In these equations, $\mu = -X^2 y'(X)$, $\rho_c/\rho_m = X^3/3\mu$; X , μ are dimensionless radius and mass, and ρ_c/ρ_m is the central-to-mean density ratio.

3. The perturbation method

Consider eq. (1) as ODE depending on parameter n , then assuming n as a small parameter, $n \ll 1$, we expand the r.s. of eq. (1) to the second order of n :

$$y = y_0 + n y_1 + n^2 y_2, \quad y^n = 1 + n \ln(y_0) + n^2 \left(\frac{y_1}{y_0} + \frac{1}{2} \ln^2(y_0) \right). \quad (5)$$

From eqs. (1, 5) we have three coupled ODEs for three functions y_0 , y_1 , y_2 :

$$\frac{1}{x^2} \frac{d}{dx} \left(x^2 \frac{dy_0}{dx} \right) = -1, \quad y_0(0) = 1, \quad y_0'(0) = 0; \quad (6)$$

$$\frac{1}{x^2} \frac{d}{dx} \left(x^2 \frac{dy_1}{dx} \right) = -\ln(y_0), \quad y_1(0) = y_1'(0) = 0; \quad (7)$$

$$\frac{1}{x^2} \frac{d}{dx} \left(x^2 \frac{dy_2}{dx} \right) = -\left(\frac{y_1}{y_0} + \frac{1}{2} \ln^2(y_0) \right), \quad y_2(0) = y_2'(0) = 0. \quad (8)$$

Initial conditions in eqs. (6, 7, 8) are defined by the form of series expansion of the solution of LEE of arbitrary n at $x = 0$:

$$y = 1 - \frac{1}{6} x^2 + \frac{n}{5!} x^4 + \frac{n(8n-5)}{3 \cdot 7!} x^6 + \frac{n(122n^2 - 183n + 70)}{9 \cdot 9!} x^8 + \dots \quad (9)$$

Expanding eq. (9) to the second order of n , we have the series expansions for functions y_1, y_2 at $x = 0$:

$$y_1 = \frac{1}{5!} x^4 + \frac{5}{3 \cdot 7!} x^6 + \frac{70}{9 \cdot 9!} x^8 + \dots, \quad (10)$$

$$y_2 = -\frac{8}{3 \cdot 7!} x^6 - \frac{183}{9 \cdot 9!} x^8 - \dots \quad (11)$$

Note that $y_1 > 0$ and $y_2 < 0$.

Before solving eqs. (7, 8) I'd like to dwell some on the validity of the approach used. There are some more or less rigorous theorems in mathematical analysis about dependence of ODE's solutions on parameter (see e.g. Korn & Korn (1968), sect. 10.2-7c, and Kamke (1959), sect. 2.5., SK78 and references therein) and this note is not the right place to discuss them, see e.g. SK78. In our case, we loosely reduce these theorems to assertion that, if for $n = 0$ and $n \ll 1$ the solutions of ODE exist, then the formal expansion of ODE and its solution in the above mentioned way converge to the correct solution (actually, this is rewording of the Poincaré-Lyapunov theorem, see SK78 and references therein). In our case of LEE we "do know" that desired solution exists (and we may find it e.g. by the numerical integration of ODE) therefore we may conclude that the procedure should give (in principle - for infinite number of terms in eq. (5)!) correct solution. Does the particular truncated approximation give correct (at least in the numerical sense) solution - this is not guaranteed and should be checked each time separately.

As it happens in our case, one meets no trouble with the perturbation method in the first approximation, to first order of n (see however sect. 7), while already in the second approximation there are some difficulties.

4. First approximation

First approximation (zero'th approximation coincides with LEE of index $n = 0$ and needs no additional solving) was solved in SK78 and we present here results with some

additional formulas needed for the second approximation.

We have from eq. (6) (I remind that the shorthand $s \equiv \sqrt{6}$ is used):

$$y_0(x) = 1 - \frac{1}{6} x^2; \quad y_0(s) = 0; \quad y_0'(s) = -\frac{1}{3} s; \quad y_0''(s) = -\frac{1}{3}. \quad (12)$$

Using eqs. (6, 12) we get from eq. (7):

$$y_1'(x) = -\frac{1}{x^2} \int_0^x t^2 \ln(y_0(t)) dt, \quad (13)$$

or

$$y_1'(x) = \frac{4}{x} + \frac{2}{9} x - \frac{1}{x} \ln(y_0) - \frac{2s}{x^2} \operatorname{arctanh}\left(\frac{x}{s}\right). \quad (14)$$

We need also value of y_1' at $x = s$:

$$y_1'(s) = \frac{2}{9} s (4 - 3 \ln 2). \quad (15)$$

Using eq. (14) we get from eq. (8):

$$y_1(x) = \int_0^x y_1'(t) dt, \quad (16)$$

or

$$y_1(x) = \frac{5x^2}{18} - 4 + (2 + y_0) \ln(y_0) + 4 \frac{s}{x} \operatorname{arctanh}\left(\frac{x}{s}\right), \quad (17)$$

$$y_1(s) = 4 \ln 2 - \frac{7}{3}. \quad (18)$$

5. Second approximation

We get from eq. (8):

$$y_2'(x) = -\frac{1}{x^2} \int_0^x t^2 \left[\frac{y_1(t)}{y_0(t)} + \frac{1}{2} \ln^2(y_0(t)) \right] dt. \quad (19)$$

Using $y_0(t)$ from eq. (12) and $y_1(t)$ from eq. (17) we can integrate eq. (19) analytically however expression is cumbersome and we do not present it here. We only mention that $y_2'(x)$ has a log-singularity at $x = s$:

$$\lim_{x \rightarrow s} y_2'(x) = -\frac{1}{6} \int 6 \frac{y_1(s)}{2(1 - x/s)} dx = \frac{s}{2} y_1(s) \ln(1 - \frac{x}{s}). \quad (20)$$

Using solution $y_2'(x)$ we get further:

$$y_2(x) = \int_0^x y_2'(t) dt. \quad (21)$$

Again analytical solution is very cumbersome and we present only the value of $y_2(x)$ at $x = s$:

$$y_2(s) = \frac{1}{9} (413 - 21 \pi^2 - 402 \ln(2) + 144 \ln^2(2)). \quad (22)$$

6. Series solution

In SK77 the series solution was given for LEE of arbitrary n :

$$y(x) = 1 + \sum_{k=1}^{\infty} a_k x^{2k}, \quad (23)$$

with recursion relation between coefficients a_k :

$$a_{m+1} = \frac{1}{m(m+1)(2m+3)} \sum_{i=1}^m (i n + i - m)(m - i + 1) (3 + 2(m - i)) a_i a_{m+1-i}, \quad (24)$$

where $m \geq 1$, and $a_1 = -1/6$. This recursion relation is valid for any n . The important problem of the convergence radius r of the series (23, 24) can not be solved in the general case.

We consider here the interesting case of small values of n . In the linear approximation, we have for the first several coefficients:

$$a_1 = -\frac{1}{3!}, \quad a_2 = \frac{n}{5!}, \quad a_3 = \frac{5n}{3 \cdot 7!}, \quad a_4 = \frac{70n}{9 \cdot 9!}, \quad a_5 = \frac{3150n}{45 \cdot 11!}. \quad (25)$$

We note that starting with a_2 all coefficients are positive. To derive a general formula for coefficients (in the same approximation, to the first order of n), we retain in the sum in r.s. of the recursion relation (24) only first (with $i = 1$) and last (with $i = m$) terms, and we get the following relation:

$$a_{m+1} = \frac{(m-1)(2m+1)}{6(m+1)(2m+3)} a_m, \quad m \geq 2, \quad a_1 = -\frac{1}{6}, \quad a_2 = \frac{n}{5!}. \quad (26)$$

From this we get for the convergence radius of the series solution of the LEE for quasi-incompressible planet:

$$r = \lim_{m \rightarrow \infty} \sqrt{\frac{a_m}{a_{m+1}}} = s. \quad (27)$$

7. Density and pressure distribution

Besides solving LEE (which is of its own interest and import) it's interesting to look for distribution of some physical parameters, namely density and pressure over the model's volume. We mention that under the gravitationally equilibrium condition, the pressure distribution is more strictly defined: the pressure should be a *continuous* function of radius (while density can have *discontinuity* - as in the first order phase transition) and the pressure should be monotonically decreasing.

Consider the spherical polytrope of index n , with central pressure P_c , and central density ρ_c . Transition from the "physical" current radius (distance from the center), r , to the "dimensional" radius x (which appears in all equations above) is as follows:

$$r = \alpha \cdot x, \quad \alpha = \left[\frac{(n+1) P_c}{4 \pi G \rho_c^2} \right]^{1/2}, \quad (28)$$

where G is the gravitational constant.

Pressure $P(r)$ and density $\rho(r)$ at distance r are given by:

$$P(r)/P_c = y^{1+n}(x), \quad \rho(r)/\rho_c = y^n(x), \quad (29)$$

where $y(x)$ is the function which appears in all equations above.

In the incompressible planet (polytropic index $n = 0$), the density ρ is constant over the volume, while the pressure decreases as:

$$P(r) = P_c \cdot y_0(x) = P_c \cdot (1 - (x/s)^2). \quad (30)$$

In the quasi-incompressible planet, in the first approximation (linear in n) for density as a function of radius we have:

$$\rho(r)/\rho_c = y^n(x) = 1 + n \ln(y_0(x)) = 1 + n \ln(1 - (x/s)^2), \quad (31)$$

and at $x = s$ (inside the planet close to its surface) there is a log-singularity of density. This is the reducible singularity, that is by integrating over volume we get correct values of radius, mass, and central-to-mean density ratio. Still this singularity in the density distribution is the unpleasant and non-physical characteristic of the perturbation method. Sure, in reality there is no density singularity inside the quasi-incompressible planet, and we may avoid it in our calculation keeping n in power of $y(x)$ and rewrite eq. (31) as follows:

$$\rho(r)/\rho_c = y^n(x) = [y_0(x) + n y_1(x)]^n, \quad (32)$$

or even more accurately:

$$\rho(r)/\rho_c = y^n(x) = [y_0(x) + n y_1(x) + n^2 y_2(x)]^n. \quad (33)$$

As to the pressure distribution, there is no difficulty because in the linear approximation we have:

$$P(r)/P_c = y^{n+1}(x) = y_0(x) + n [y_1(x) + y_0(x) \ln(y_0(x))], \quad (34)$$

and there is no singularity, in the pressure distribution, inside the model (while there is a log-singularity in *derivative* dP/dr which is proportional to the density - in accordance with eq. (31)).

Again, as for the density distribution, we may write more accurately:

$$P(r)/P_c = [y_0(x) + n y_1(x) + n^2 y_2(x)]^{n+1}, \quad (35)$$

avoiding the singularity in the derivative dP/dr .

8. Global parameters

Now we are ready to find some global parameters of the quasi-incompressible planet.

8.1. Radius

According to eq. (28), the total radius R of the polytrope is

$$R = \alpha \cdot X, \quad (36)$$

where the first zero X is such that:

$$y(X) = y_0(X) + n y_1(X) + n^2 y_2(X) = 0, \quad (37)$$

$$X = s + n \delta_1 + n^2 \delta_2, \quad (38)$$

with δ_1, δ_2 still to be found.

In the spirit of the perturbation method, we should keep, in eq. (37), the argument X in $y_0(X)$ to the second order of n , in $y_1(X)$, to the first order of n , and in $y_2(X)$, only $X = s$. Expanding all functions to the second order of n , combining terms with the same power of n , equalizing coefficients of n and n^2 to zero, we get system of two equations for δ_1 and δ_2 , and solving these equations we get finally:

$$\delta_1 = \frac{3}{s} y_1(s) = \frac{1}{s} (12 \ln 2 - 7); \quad (39)$$

$$\delta_2 = \frac{3}{s} \left(y_2(s) + \frac{3}{s} y_1(s) y_1'(s) - \frac{1}{4} y_1^2(s) \right), \quad (40)$$

$$\delta_2 = \frac{1}{12s} (1379 - 84\pi^2 - 888 \ln 2 + 144 \ln^2 2). \quad (41)$$

From eqs. (38, 39, 41) we have numerically:

$$X = 2.44948974278 + 0.537975784794n + 0.12328309n^2. \quad (42)$$

Note that X (and hence radius R) is the only parameter of the quasi-incompressible planet which can be calculated to the second order of n . Due to the singularity of $y_2'(s)$, the second approximation is not possible for other global parameters of the quasi-incompressible planet.

8.1.1. Comparison with the numerical values

For $n = 1/2$, numerically $X_{num} = 2.752698$, analytically, from eq. (42) $X = 2.749298$ with relative difference of about .12%.

For $n = 1/10$, numerically $X_{num} = 2.504545$, analytically $X = 2.50452015$ with relative difference of .01%.

For $n = 1/1000$, numerically $X_{num} = 2.45004$, analytically $X = 2.45002978$ with difference in the last digit of X_{num} .

We mention that the cases of very small values of n are more easier (and more accurately) described by analytical formulas than by numerical calculations.

Besides, no numerical calculation can give analytical formula for $X(n)$ even in the linear approximation in n .

8.2. Mass

The running mass inside the polytropic sphere of radius r is:

$$m(r) = 4\pi\rho_c\alpha^3[-x^2y'(x)], \quad (43)$$

and the total mass of the polytrope is:

$$M = 4\pi\rho_c\alpha^3\mu, \quad \mu = -X^2y'(X). \quad (44)$$

In order to calculate $\mu(X)$ to the second order of n , we need $y_2'(s)$ which is infinite (see eq. (20)), so we restrict ourselves by the linear approximation, already considered in SK78. By the same way as in section (8.1), we get:

$$\mu = 2s + 6n \left[\delta_1 - y_1'(s) \right] = 2s [1 - (37/6 - 8 \ln 2)n] = 2s (1 - 0.62148922n). \quad (45)$$

8.3. Central-to-mean density ratio

For the polytrope, from eqs. (36, 44) we have for the central-to-mean density ratio:

$$\frac{\rho_c}{\rho_m} = \frac{X^3}{3\mu}, \quad (46)$$

and though we calculated X in the second approximation, however the parameter μ was calculated only in the first approximation, therefore we can calculate the central-to-mean density ratio for the quasi-incompressible planet only in the first approximation, already considered in SK78:

$$\frac{\rho_c}{\rho_m} = 1 + \left(\frac{8}{3} - 2 \ln 2 \right) n = 1 + 1.280372 n. \quad (47)$$

8.4. Moment of inertia

The central moment of inertia of the spherical polytrope is:

$$I = \int_0^M r^2 dm = k_I M R^2, \quad k_I = \frac{1}{\mu X^2} \int_0^X y^n(x) x^4 dx, \quad (48)$$

and to the first order of n we get:

$$k_I = \frac{3}{5} - \frac{6n}{25}. \quad (49)$$

8.5. Milne integral

In the theory of polytropes, there is Milne's relation (Milne 1929) which also appeared in the "refined" theory of rotating polytropes ((Chandrasekhar & Lebovitz 1962), eq. (87)):

$$MI = \int_0^X y^{1+n}(x) x^2 dx = \frac{1+n}{5-n} X^3 \left[y'(X) \right]^2. \quad (50)$$

Using our formulas we find for both sides of the equation (50), for the quasi-incompressible planet ($n \ll 1$):

$$MI = \frac{4}{5} s + \frac{2}{25} s (120 \ln 2 - 79) n. \quad (51)$$

We can not calculate MI in the second approximation because this requires the $y_2'(s)$ (see r.s. of eq. (50)) which diverges (see eq. (20)).

By the way, the case of $n = 5$ in eq. (50) is interesting as well. Assuming $(5 - n)0$ as a

small parameter, expanding both sides of eq. (50) in series we find the dependence of $X(n)$, already found in SK78:

$$X(n) = \frac{32\sqrt{3}}{\pi} \frac{1}{5-n}, \quad 0 < 5-n \ll 1. \quad (52)$$

8.6. One interesting relation

In (Seidov & Skvirsky 2000), the interesting relation is introduced between the gravitational potential energy, W , the central potential, U_c , and the mass of the celestial body. We confine ourselves here by the case of spherical polytropes.

From the theory of polytropes, the central gravitational potential, U_c , the central pressure P_c , and the central density, ρ_c , are (see (Chandrasekhar 1957), p. 100, eq. (85); p. 99, eq. (80,81); p. 78, eq. (99)):

$$U_c = (1+n) \frac{P_c}{\rho_c} + \frac{GM}{R}, \quad (53)$$

$$P_c = \frac{1}{4\pi(1+n)} \frac{GM^2}{R^4}, \quad (54)$$

$$\rho_c = \frac{1}{3} \frac{X}{[-y'(X)]} \rho_m = \frac{X}{[-y'(X)]} \frac{M}{4\pi R^3}. \quad (55)$$

Combining eqs. (53, 54, 55), we get for the central gravitational potential of the spherical polytrope:

$$U_c = \left(1 + \frac{1}{[-X y'(X)]}\right) \frac{GM}{R}. \quad (56)$$

Potential energy of polytrope (we loosely take positive sign) is ((Chandrasekhar 1957), p. 101, eq. (90)):

$$W = \frac{3}{5-n} \frac{GM^2}{R}. \quad (57)$$

Now we have, for polytropes, the WUM -ratio introduced in (Seidov & Skvirsky 2000):

$$WUM = \frac{W}{U_c M} = \frac{3}{5-n} \frac{1}{1+X/\mu}. \quad (58)$$

Note in brackets that WUM -ratio is remarkably constant ($=2/5$) for the rotating triaxial homogeneous ellipsoids (see (Seidov & Skvirsky 2000)). In our case of the quasi-incompressible planet, using eqs. (38, 39, 45, 58) we have to the first order of n :

$$WUM = \frac{2}{5} \left[1 - n \left(\frac{22}{15} - 2 \ln 2\right)\right]. \quad (59)$$

8.7. Parameter ω

In (Christensen-Dalsgaard & Mullan 1994), the parameter ω for the polytrope was introduced (see their eq. (A14)):

$$\omega = \frac{1}{4\pi(1+n)} \left(\frac{3}{X} \right)^{1+1/n} \left[-y'(X) \right]^{1-1/n}, \quad (60)$$

with a note that at $n = 0$ this expression is undefined, and from numerical calculations, at $n = 0$, $\omega \approx 0.033175949$. This is a good chance to demonstrate usefulness of the perturbation method by SK: using the formulas of the previous sections and omitting some algebra we write down the final expression for ω to the first order of n :

$$\omega = \frac{3}{2\pi} \exp(-8/3) \left\{ 1 - n \left[1 + \frac{1}{2} \left(\frac{8}{3} - \ln 6 \right)^2 \right] \right\}, \quad (61)$$

or numerically

$$\omega = 0.033175904175 (1. - 1.382731302 n),$$

and we note again that the analytical expression helps to check the numerical calculations (and the last two digits of the "numerical" value of ω at $n = 0$ differ from the "analytical" value).

9. "Numerical" perturbation method

Though being slightly out-of-topic, we mention that the perturbation method can be used in its numerical modification as well.

In the case of polytropes, it is particularly interesting to do this for the case $n = 3$. We briefly describe the method and give the short results in the first approximation. We take in the r.s. of eq. (1), $n = 3 - \delta$ with $0 < \delta \ll 3$, then we have

$$(y_0 - \delta y_1)^{3-\delta} = y_0^3 - y_0^2 (3 y_1 + y_0 \ln y_0) \delta. \quad (62)$$

Here y_0 is solution of LEE of index $n = 3$ and y_1 is the "perturbation" function. Also, because $\delta > 0$, zero X of function $y_0 - \delta y_1$ is *less* than X_0 , zero of function y_0 . Due to an inequality $X < X_0$ there is no problems with behavior of functions near boundary as in the quasi-incompressible case. As in the text above, we have system of two ODE's (each of them of the second order) for functions y_0 and y_1 . We can solve this system numerically and find all relevant functions y_0 , y_1 , y_0' , and y_1' and particularly their values at $x = X_0$. Then we can

get the formulas similar to ones in section (8).

We present the results of the numerical calculations:

$$X_0 = 6.8968498, \quad y'_0(X_0) = -0.0424297317, \quad (63)$$

$$y_1(X_0) = 0.16547670449, \quad y'_1(X_0) = -0.00616735. \quad (64)$$

From this we have for the dimensionless radius of polytrope with index n close to 3:

$$X = 6.896848 + (3 - n) y_1(X_0)/y'_0(X_0) = 6.896848 - 3.90002(3 - n). \quad (65)$$

Numerically, at $n = 3 - 0.1$, $X = 6.526374$, while from the linear perturbation method (65): $X = 6.5068$.

Also, numerically, at $n = 3 + 0.1$, $X = 7.308484$, while from the linear perturbation method (65): $X = 7.2868$.

Deviations of numerical values in two cases correspond (correctly) to the positive second derivative of function $X(n)$.

10. Summary

In this note we present the exact analytical solutions for the internal structure and global parameters of the quasi-incompressible planet modeled as the polytrope of small index $0 < n \ll 1$. The perturbation method used here is not rigorously justified by means of the theory of differential equations and there are some problems about application of the method in the interval of argument where the perturbation function is of the same order or even larger than the initial non-perturbed function. The problem of justification is here similar to the problem of rotationally distorted polytropes which was already discussed some in the astrophysical literature (see e.g. Chandrasekhar & Lebovitz (1962)). Still validity of any method in applicational sciences (as astrophysics is such relative to mathematics) may be compared by numerical calculation and (astro)-physical "common sense" and I only hope that this note may trigger some discussion as well.

11. Acknowledgments

This paper is to be devoted to the memory of my dear friend and PhD student late Rafael Khejrullaevich Kuzakhmedov (=Rafik) who having been a very strong theor. physicist flatly rejected to admit any significance in our two minor notes (SK77, SK78) and had never got even PhD degree. Only my vision of astrophysics and Rafik's incredible ability of what is

called now symbolic mathematics (and my long pressure on Rafik!) could eventually produce these two short notes which I count among my best ones.

Also the financial support from the Israeli Ministry of Science and the Administration of the College of Judea and Samaria is duly acknowledged.

REFERENCES

- Caimmi, R. 1987, *Ap&SS*, 140, 1
- Chandrasekhar, S. 1957, *An Introduction to the Study of Stellar Structure* (Chicago: Dover)
- Chandrasekhar, S., & Lebovitz, N. R. 1962, *ApJ*, 136, 1082
- Christensen-Dalsgaard, J., & Mullan, D. J. 1994, *MNRAS*, 270, 921
- Horedt, G. P. 1987, *Astron. Ap.*, 172, 359
- Horedt, G. P. 1990, *ApJ*, 357, 560
- Jabbar, J. R. 1984, *Ap&SS*, 100, 447
- Kamke, E. 1959, *Differentialgleichunge* (6th ed., Leipzig)
- Korn, G. A. & Korn, T. M. 1968, *Mathematical Handbook* (2nd ed., New York: McGraw-Hill)
- Medvedev, M. V., & Rybicki, G. 2001, *ApJ*, 555, 863
- Milne, E. A. 1929, *MNRAS*, 89, 739
- Mohan, C. & Al-Bayaty, A. R. 1980, *Ap&SS*, 73, 227
- Seidov, Z. F. 1978, *Sov. Astron. Lett.*, 4(3), 144
- Seidov, Z. F. 1978, in *Sources of Gravitational Radiation*, Proceed. Batelle Seattle Workshop, ed. L.L. Smarr (New York: CUP), 480
- Seidov, Z. F. 1979, *Dokl. AN AzSSR*, 35, no. 1, 21
- Seidov, Z. F. 1979, in *IAU Coll. no. 53*, Rochester, (pp. 478-482)
- Seidov, Z. F., & Kuzakhmedov, R. Kh. 1977, *Sov. Astron.*, 21, 399 (SK77)
- Seidov, Z. F., & Kuzakhmedov, R. Kh. 1978, *Sov. Astron.*, 22, 711 (SK78)
- Seidov, Z. F., Sharma, J. P., & Kuzakhmedov, R. Kh. 1979, *Dokl. AN AzSSR*, 35, no. 5, 21; no. 6, 25
- Seidov, Z. F., & Skvirsky, P. I. 2000, preprint (astro-ph/0003064)